# The Representation Technique
# Cryptanalysis for Dlog, SubsetSum, Decoding


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# Discrete Logarithms

## DLP: Discrete Logarithm Problem

**Given:** Generator $g$ for $G = \langle g \rangle$ with $2^{n-1} \leq |G| < 2^n$, $\beta = g^x$
**Find:** $x = \mathrm{dlog}_g \beta \in \mathbb{Z}_{|G|}$

**Examples:**

- $G = (\mathbb{Z}, +) = \langle 1 \rangle$, $x = \mathrm{dlog}_1 \beta = \beta$
- $G = (E(\mathbb{F}_p), +)$, best algorithm $\tilde{\mathcal{O}}(\sqrt{|G|}) = \tilde{\mathcal{O}}(2^{\frac{n}{2}})$.
- $G = (\mathbb{Z}_p^*, \cdot)$, best algorithm sub-exponential
- $G$ generic: $\Omega(\sqrt{|G|})$

**Variants:** small $x$, small Hamming weight $x$, faulty $x$, many $x$

# DLP Enumeration

**Algorithm** Brute-Force DLP

**Input:** $g, \beta$

1. $x = 0$.
2. **While** $(g^x \neq \beta)$ **do** $x = x + 1$;

**Output:** $x = \mathrm{dlog}_g \beta$

**Runtime:**

- Need $x$ iterations of while-loop, each costs one group operation.
- $\mathcal{O}(x) = \mathcal{O}(|G|) = \mathcal{O}(2^n)$ group operations.
- Each group operation usually costs $\mathcal{O}(\log^c n)$ bit operations.
- **Notice:** Brute-Force not so bad for small $x$.

# Reaching Square Root Complexity

**Idea**:

- Write $x = x_1 + x_2 2^{n/2}$ with $0 \leq x_1, x_2 < 2^{n/2}$.
- Use identity $g^{x_1} = \beta \cdot (g^{-2^{\frac{n}{2}}})^{x_2}$.

**Algorithm** Meet-in-the-Middle DLP

**Input:** $g, \beta$

1. **For** $0 \leq i < 2^{n/2}$ **do** store $(i, g^i)$ in list $L$.
2. Sort list $L$ according to second entry.
3. **For** $0 \leq j < 2^{n/2}$ **do** if $\exists (i, \beta \cdot (g^{-2^{\frac{n}{2}}})^j) \in L$, output $x = i + j2^{n/2}$.

**Output:** $x = \mathrm{dlog}_g \beta$

**Correctness:** MitM terminates iff $(i, j) = (x_1, x_2)$.
**Run time:** $\tilde{\mathcal{O}}(2^{\frac{n}{2}}) = \tilde{\mathcal{O}}(\sqrt{|G|})$. But also memory $\tilde{\Theta}(\sqrt{|G|})$.

**Exercise:** Modify MitM such that it has runtime $\tilde{\mathcal{O}}(\sqrt{x})$.

# Multiple Discrete Logarithms

## Multiple DLP

**Given:** Generator $g$ for $G = \langle g \rangle$ with $2^{n-1} \leq |G| < 2^n$,
$\beta_1 = g^{x_1}, \ldots, \beta_k = g^{x_k}$

**Find:** $x_1, \ldots, x_k$

**Easy:** $\tilde{\mathcal{O}}(k \cdot \sqrt{|G|})$.

**Exercise:** Show that Multiple DLP can be solved in $\tilde{\mathcal{O}}(\sqrt{k \cdot |G|})$.

# Small Weight Discrete Logarithms

## Small weight DLP

**Given:** Generator $g$ for $G = \langle g \rangle$ with $2^{n-1} \leq |G| < 2^n$,
$\beta = g^x$ with known Hamming weight $\mathrm{wt}(x) = \alpha n$, $\alpha \in [0, 1]$

**Find:** $x$

## **Algorithm** Brute-Force Small weight DLP

**Input:** $g, \beta$, $\alpha$

1. **For all** $x$ with $\mathrm{wt}(x) = \alpha n$ **do** if $(g^x = \beta)$ output $x$;

**Output:** $x = \mathrm{dlog}_g \beta$

**Run time:** $\tilde{\mathcal{O}}(\binom{n}{\alpha n})$. How good is that?

# Bounding Binomial Coefficients

**Theorem** Binomials

We have $\binom{n}{\alpha n} = \tilde{\Theta}(2^{H(\alpha)n})$ with $H(\alpha) = -\alpha\log(\alpha) - (1-\alpha)\log(1-\alpha)$.

By Stirling's formula $n! \sim \sqrt{2\pi n} \cdot (\frac{n}{e})^n$ we have

$$\begin{aligned}\binom{n}{\alpha n} &= \frac{n!}{(\alpha n)!((1-\alpha)n)!} = \tilde{\Theta}\left(\frac{(\frac{n}{e})^n}{(\frac{\alpha n}{e})^{\alpha n}(\frac{(1-\alpha)n}{e})^{(1-\alpha)n}}\right) \\ &= \tilde{\Theta}\left(2^{(-\alpha\log\alpha-(1-\alpha)\log(1-\alpha))n}\right) = \tilde{\Theta}(2^{H(\alpha)n})\end{aligned}$$

**Corollary**

For $0 \le \alpha \le \beta \le 1$: $\binom{\beta n}{\alpha n} = \binom{\beta n}{\alpha\frac{1}{\beta}\beta n} = \tilde{\Theta}(2^{H(\frac{\alpha}{\beta})\cdot\beta n})$.

# Small weight Discrete Logarithms

Brute-Force Small Weight DLP: $\tilde{\mathcal{O}}(\binom{n}{\alpha n}) = \tilde{\mathcal{O}}(2^{H(\alpha)n})$, $\alpha = \frac{1}{2} : \tilde{\mathcal{O}}(2^n)$.

**Exercise 1:** Assume that we get the promise $x = x_1 + x_2 2^{n/2}$ with

$$0 \leq x_1, x_2 < 2^{n/2} \text{ and } \mathrm{wt}(x_1) = \mathrm{wt}(x_2) = \alpha \cdot \frac{n}{2}.$$

Devise a MitM algorithm with run time $\tilde{\mathcal{O}}(2^{\frac{H(\alpha)}{2}n})$.

**Exercise 2:** Do Exercise 1 without promise.

# Faulty Discrete Logarithms

## Faulty DLP

**Given:** Generator $g$ for $G = \langle g \rangle$ with $2^{n-1} \leq |G| < 2^n$,
$\beta = g^x$, faulty $\tilde{x}$ with $\alpha n$, $\alpha \in [0,1]$ many $1 \to 0$-flips of $x$

**Find:** $x$

**Mini Exercise**: Show how Faulty DLP relates to Small weight DLP.

# Finding a function collision

## Collision finding

**Given:** function $f : \{0,1\}^n \rightarrow \{0,1\}^n$ (with random properties)
**Find:** $x_1 \neq x_2$ with $f(x_1) = f(x_2)$

- $\Pr_{x_1 \neq x_2}(f(x_1) = f(x_2)) = \frac{1}{2^n}$
- Brute Force: Sample $2^n$ many pairs $(x_1, x_2)$.

# Birthday Paradox – Meet in the Middle

**Algorithm** List Collision Finding

**Input:** $f : \{0,1\}^n \to \{0,1\}^n$

1. Compute list $L$ with entries $(x_i, f(x_i))$ for $i = 1, \dots, 2^{n/2} + 1$.
2. Search for $(x_i, y), (x_j, y) \in L$ with $i \neq j$.

**Output:** Collision or $\perp$

**Run time & Success probabilty:**

- Run time $\tilde{\mathcal{O}}(2^{\frac{n}{2}})$ (but also the same memory).
- $L$ does not contain a collision with probability

$$\prod_{i=0}^{2^{n/2}} \left(1 - \frac{i}{2^n}\right) \leq \prod_{i=1}^{2^{n/2}} e^{-\frac{i}{2^n}} = e^{-\sum_{i=1}^{2^{n/2}} \frac{i}{2^n}} = e^{-\frac{2^{n/2}(2^{n/2}+1)}{2 \cdot 2^n}} \leq e^{-\frac{1}{2}} \approx 0.6.$$

- Thus, we succeed with probability $\approx 0.4$.

# Iterating a function

- Consider sequence: $x, f(x), f(f(x)), f(f(f(x))), \ldots$.
- Let us use notation $f^i(x)$ for $i$ applications.
- Let $\gamma, \lambda > 0$ be minimal with $f^\gamma(x) = f^{\gamma+\lambda}(x)$. Then

$$f^{\gamma+1}(x) = f^{\gamma+\lambda+1}(x), f^{\gamma+2}(x) = f^{\gamma+\lambda+2}(x), \ldots$$

- By the argumentation before we expect that $\gamma + \lambda \approx 2^{\frac{n}{2}}$.

# Cycle Finding

**Algorithm** Cycle Finding

**Input:** $f : \{0,1\}^n \rightarrow \{0,1\}^n$

1. **Repeat** Choose start point $x \in \{0,1\}$ **until** $x \neq f(x)$.
2. Set $i = 1,\ k_i = f(x), k_{2i} = f(f(x))$.
3. **While** $k_i \neq k_{2i}$ **do**
    1. $k_{i+1} = f(k_i), k_{2(i+1)} = f(f(k_{2i}))$. Set $i = i + 1$.
4. Set $\ell = 0,\ k_\ell = x$.
5. **While** $f(k_\ell) \neq f(k_{\ell+i})$ **do** $k_{\ell+1} = f(k_\ell), k_{\ell+i+1} = f(k_{\ell+i}), \ell = \ell + 1$.

**Output:** $x_1 = k_\ell, x_2 = k_{\ell+i}$ with $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

# Cycle Finding

**Correctness**

- After the first while-loop we have $k_i = k_{2i}$.
- We already know that $k_j = k_{j+c\lambda}$, $c \in \mathbb{N}$ for all $j \geq \gamma$.
- We conclude that $i = k\lambda$.
- In the second loop we find the minimum $\gamma$ for which

$$k_\gamma = k_{\gamma+k\lambda}.$$

- At termination we have $f(k_{\gamma-1}) = f(k_{\gamma+k\lambda-1})$ which implies

$$f(x_1) = f(k_{\gamma-1}) = k_\gamma = k_{\gamma+k\lambda} = f(k_{\gamma+k\lambda-1}) = f(x_2).$$

- Furthermore, $x_1 \neq x_2$ by minimality of $\gamma$.

**Complexity**

- Memory consumption $\tilde{\mathcal{O}}(1)$.
- After $\gamma + \lambda \approx 2^{\frac{n}{2}}$ we cycle. The cycle length is $\lambda$. (**While**-loop in 3)
- In total, we need $2(\gamma + \lambda) \approx 2^{\frac{n}{2}+1}$ iterations until termination.

# Two functions

**Theorem** Rho Method

In a function $f : \{0,1\}^n \rightarrow \{0,1\}^n$ we find a collision in time $\tilde{\mathcal{O}}(2^{\frac{n}{2}})$ with space $\tilde{\mathcal{O}}(1)$.

**Two function collision finding**

**Given:** functions $f_1 : \{0,1\}^n \rightarrow \{0,1\}^n$, $f_2 : \{0,1\}^n \rightarrow \{0,1\}^n$ (with random properties)

**Find:** $x_1, x_2$ with $f_1(x_1) = f_2(x_2)$

**Theorem** Rho Method

In two functions $f_1, f_2 : \{0,1\}^n \rightarrow \{0,1\}^n$ we find a collision in time $\tilde{\mathcal{O}}(2^{\frac{n}{2}})$ with space $\tilde{\mathcal{O}}(1)$.

**Exercise:** Adapt the previous method for two functions.

# Rho Method for DLP

**Representation of DLP:** Define

$$f_1 : \mathbb{Z}_{|G|} \to G, x \mapsto g^{x_1} \text{ and } f_2 : \mathbb{Z}_{|G|} \to G, x_2 \mapsto \beta \cdot g^{-x_2}.$$

- Any collision $(x_1, x_2)$ satisfies $g^{x_1} = \beta \cdot g^{-x_2}$.
- Thus, $x = x_1 + x_2 \bmod \mathbb{Z}_{|G|}$ solves DLP.
- There exist $\mathbb{Z}_{|G|}$ many representations, respectively collisions.
- For solving DLP it suffices to find a *single* representation $(x_1, x_2)$.

**Definition** Representation

Let $x = x_1 + x_2$. Then $(x_1, x_2)$ is called a *representation* of $x$.

**Theorem** Rho Method for DLP

DLP can be solved in any group $G$ in time $\tilde{\mathcal{O}}(\sqrt{|G|})$ and memory $\tilde{\mathcal{O}}(1)$.

**Exercise:** Show an $\tilde{\mathcal{O}}(x^{\frac{3}{2}})$-algorithm for small $x$-DLP with memory $\tilde{\mathcal{O}}(1)$.

# Small Weight DPL with Low Memory

**Promise:** $x = x_1 + x_2 2^{n/2}$ with $0 \le x_i < 2^{n/2}$ and $\mathrm{wt}(x_i) = \alpha \cdot \frac{n}{2}$.

- Search space $\mathcal{S} = \{x_i \in \mathbb{Z}_{2^{n/2}} \mid \mathrm{wt}(x_i) = \alpha \cdot \frac{n}{2}\}$.
- Therefore $|\mathcal{S}| = \binom{n/2}{\alpha \cdot n/2} = \tilde{\Theta}(2^{H(\alpha)n/2})$.
- Let $h : G \to \mathcal{S}$. Define $f_i : \mathcal{S} \to \mathcal{S}$ with

$$x_1 \mapsto h(g^{x_1}) \text{ and } x_2 \mapsto h(\beta \cdot g^{-x_2 2^{n/2}}).$$

**Algorithm** Folklore Low Weight DPL with Low Memory

**Input:** $f_1, f_2, h$

1. **Repeat**
    1. Find a random collision $(x_1, x_2)$ in $f_1, f_2$
2. **Until** $g^{x_1} = \beta \cdot g^{-x_2 2^{n/2}}$

**Output:** $x = x_1 + x_2 2^{n/2}$

# 0.75 Algorithm

**Run Time:**

- Every iteration costs $\tilde{\mathcal{O}}(\sqrt{|S|})$.
- Since $f_i : \mathcal{S} \rightarrow \mathcal{S}$, we expect $|\mathcal{S}|$ collisions.
- $x$ has a unique representation as $x = x_1 + x_2 2^{\frac{n}{2}}$.
- Therefore only a single collisions $(x_1, x_2)$ satisfies $g^{x_1} = \beta \cdot g^{-x_2 2^{n/2}}$.
- The probability that an iteration succeeds is thus

$$p = \Pr[(x_1, x_2) \text{ satisfies } g^{x_1} = \beta \cdot g^{-x_2}] = \frac{1}{|\mathcal{S}|}.$$

- We obtain expected run time

$$p^{-1}\tilde{\mathcal{O}}(\sqrt{|S|}) = \tilde{\mathcal{O}}(|S|^{\frac{3}{2}}) = \tilde{\mathcal{O}}(2^{\frac{3}{4}H(\alpha)n})$$

- For $\alpha = \frac{1}{2}$ this is time $2^{\frac{3}{4}n}$ as opposed to $2^{\frac{1}{2}n}$ for Rho.

# Improving a bit

**Idea:** Take the representation $x = x_1 + x_2$ with $x_1, x_2 \in \mathbb{Z}_{|G|}$ as in Rho.

- We choose $\mathrm{wt}(x_1) = \mathrm{wt}(x_2) = \frac{\alpha}{2} n$.
- Search space $\mathcal{S} = \{x_i \in \mathbb{Z}_{|G|} \mid \mathrm{wt}(x_i) = \frac{\alpha}{2} \cdot n\}$.
- Therefore $|\mathcal{S}| = \binom{n}{\alpha/2 \cdot n} = \tilde{\Theta}(2^{H(\alpha/2)n})$.
- Let $h : G \to \mathcal{S}$. Define $f_i : \mathcal{S} \to \mathcal{S}$ with

$$x_1 \mapsto h(g^{x_1}) \text{ and } x_2 \mapsto h(\beta \cdot g^{-x_2}).$$

**Algorithm** Improved Low Weight DPL with Low Memory

**Input:** $f_1, f_2, h$

1. **Repeat**
   1. Find a random collision $(x_1, x_2)$ in $f_1, f_2$
2. **Until** $g^{x_1} = \beta \cdot g^{-x_2}$

**Output:** $x = x_1 + x_2 2^{n/2}$

# 0.72 Algorithm

**Run Time:**

- Every iteration cost $\tilde{\mathcal{O}}(\sqrt{|S|})$.
- Since $f_i : \mathcal{S} \rightarrow \mathcal{S}$, we expect $|\mathcal{S}|$ collisions.
- $x$ has $\binom{\alpha n}{\frac{\alpha}{2} n} = \tilde{\Theta}(2^{\alpha n})$ many representation as $x = x_1 + x_2$.
- All representations $(x_1, x_2)$ satisfy $g^{x_1} = \beta \cdot g^{-x_2}$.
- The probability that an iteration succeeds is thus

$$p = \Pr[(x_1, x_2) \text{ satisfies } g^{x_1} = \beta \cdot g^{-x_2}] = \frac{\tilde{\Theta}(2^{\alpha n})}{|\mathcal{S}|}.$$

- We obtain expected run time

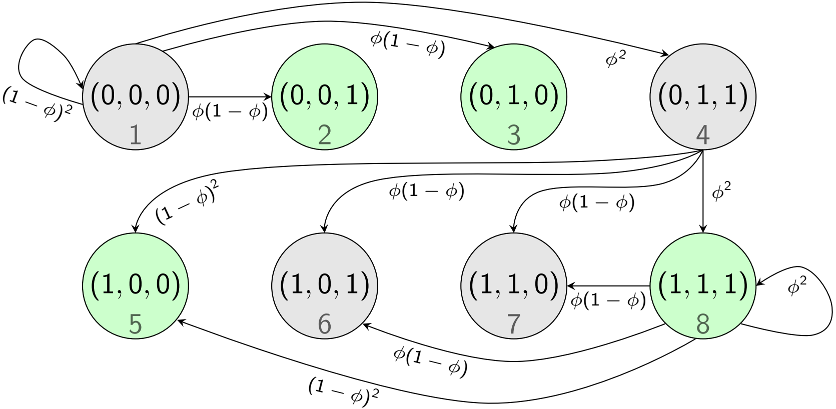
$$p^{-1} \tilde{\mathcal{O}}\left(\sqrt{|S|}\right) = \tilde{\mathcal{O}}\left(\frac{|S|^{\frac{3}{2}}}{2^{\alpha n}}\right) = \tilde{\mathcal{O}}(2^{(\frac{3}{2} H(\alpha/2) - \alpha) n})$$

- For $\alpha = \frac{1}{2}$ this is time $2^{0.72n}$ as opposed to $2^{\frac{1}{2} n}$ for Rho.

# Improving a bit more via carries

**Idea:** Take $\mathrm{wt}(x_1) = \mathrm{wt}(x_2) = \phi n \geq \frac{\alpha}{2} n$ such that $\mathrm{wt}(x_1 + x_2) = \alpha n$.

- Search space $\mathcal{S} = \{x_i \in \mathbb{Z}_{|G|} \mid \mathrm{wt}(x_i) = \phi n\}$.
- Therefore $|\mathcal{S}| = \binom{n}{\phi n} = \tilde{\Theta}(2^{H(\phi)n})$.
- Analysis: Take each 1-coordinate in $x_1, x_2$ with probability $\phi$.



# Analysis

- Define matrix $M$ for Markov process.
- Process has a stationary distribution $\pi = (\pi_1, \ldots, \pi_8)$ with $\pi = M\pi$.
- We solve the system of linear equations

$$\pi = M\pi,\ \pi_1 + \ldots + \pi_8 = 1,\ \pi_2 + \pi_3 + \pi_5 + \pi_8 = \alpha.$$

- Obtain $\alpha = 4\phi^4 - 4\phi^3 - \phi^2 + 2\phi$. Check: $\phi = \frac{1}{2} \Rightarrow \alpha = \frac{1}{2}$.
- Number of representations $(x_1, x_2)$: heuristically $\frac{|\mathcal{S}|^2}{\binom{n}{\alpha n}}$.
- This implies $p = \frac{|\mathcal{S}|}{\binom{n}{\alpha n}}$ and run time

$$p^{-1}|\mathcal{S}|^{\frac{1}{2}} = \frac{\binom{n}{\alpha n}}{|\mathcal{S}|^{\frac{1}{2}}} = 2^{(H(\alpha) - \frac{1}{2}H(\phi))n}.$$

- $\alpha = \frac{1}{2}$: Complexity $2^{\frac{1}{2}n}$.

# Parallel Collision Search

**Theorem** PCS

Given functions $f_0, \ldots, f_k : \{0,1\}^n \to \{0,1\}^n$. We find a collision between $f_0$ and all other $f_1, \ldots, f_k$ in time $\tilde{\mathcal{O}}(\sqrt{k} 2^{\frac{n}{2}})$ with space $\tilde{\mathcal{O}}(k)$.

**Multiple Dlog**

- Let $\beta_1 = g^{x_1}, \ldots, \beta_k = g^{x_k}$. Define functions $\mathbb{Z}_{|G|} \to G$ with

$$f_0 : x_0 \mapsto g^{x_0} \text{ and } f_i : x_i \mapsto \beta_i \cdot g^{-x_i} \text{ for } i = 1, \ldots, k.$$

- Collision $(x_0, x_i)$ solves $i^{th}$ dlog instance.
- Run time is $\tilde{\mathcal{O}}(\sqrt{k|G|})$ with space only $\tilde{\mathcal{O}}(k)$.

# Subset Sum

**Problem** Subset Sum

**Given:** $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{2^n}^n$ , $t \in \mathbb{Z}_{2^n}$

**Find:** $\mathbf{e} = (e_1, \ldots, e_n) \in \{0,1\}^n$ with $\sum_{i=1}^n e_i a_i = t \bmod 2^n$

**Cryptanalysis basics:**

- Brute Force: $\tilde{\mathcal{O}}(2^n)$
- Meet-in-the-Middle: $\tilde{\mathcal{O}}(2^{\frac{n}{2}})$
- **Questions:** Low Memory Algorithms? Faster?

**Exercise:**

Express DLP as a subset sum problem in $(G, \cdot)$ instead of $(\mathbb{Z}_{2^n}, +)$.

# Schroeppel-Shamir algorithm (1979)

**Idea:** Write $\sum_{i=1}^{\frac{n}{4}} e_i a_i + \sum_{i=\frac{n}{4}+1}^{\frac{1}{2}n} e_i a_i = t - \sum_{i=\frac{3}{4}n}^{\frac{1}{2}n} e_i a_i - \sum_{i=\frac{3}{4}n+1}^{n} e_i a_i$.

**Algorithm** 4-List algorithm

**Input:**

1. Generate lists $L_1, \ldots, L_4$ with
   $L_1 = \{\sum_{i=1}^{\frac{n}{4}} e_i a_i \mid (e_1, \ldots, e_{\frac{n}{4}}) \in \{0,1\}^{\frac{n}{4}}\}$, etc.
2. **Repeat**
   1. Choose $r \in_R \mathbb{Z}_{2^{\frac{n}{4}}}$.
   2. Compute $L_{12} = L_1 \bowtie_{\frac{n}{4}} L_2 := \{\sum_{i=1}^{\frac{n}{2}} e_i a_i \mid \sum_{i=1}^{\frac{n}{2}} e_i a_i = r \bmod 2^{\frac{n}{4}}\}$ and
      $L_{34} = L_3 \bowtie_{\frac{n}{4}} L_4 := \{t - \sum_{i=\frac{n}{2}+1}^{n} e_i a_i \mid t - \sum_{i=\frac{n}{2}+1}^{n} e_i a_i = r \bmod 2^{\frac{n}{4}}\}$.
   3. Compute $L = L_{12} \bowtie_n L_{34} := \{\sum_{i=1}^{n} e_i a_i \mid \sum_{i=1}^{n} e_i a_i = t \bmod 2^n\}$
3. **Until** $|L| \neq \emptyset$

**Output:** $\mathbf{e}$ from $L$

# Analysis Shamir-Shroeppel

**Correctness (Termination):**

- Let $\mathbf{e}$ be a subset sum solution. Let $r = \sum_{i=1}^{n/2} e_i a_i \bmod 2^{n/4}$.
- Assume that we choose $r$ in Step 2.2.
- Then our algorithm terminates with output $\mathbf{e}$.

**Run time:**

- Each iteration costs on expectation $\tilde{\mathcal{O}}(2^{n/4})$ time/memory.
- On expectation, it takes $2^{n/4}$ iterations for finding $r$.

**Question:** Is there a $\tilde{\mathcal{O}}(1)$ memory algorithm faster than brute-force?

# 0.75 Subset Sum

**Idea:** Use collision finding in $f_1, f_2 : \{0,1\}^{\frac{n}{2}} \to \mathbb{Z}_{2^{\frac{n}{2}}}$ with

$$f_1 :(e_1, \ldots, e_{n/2}) \mapsto \sum_{i=1}^{n/2} e_i a_i \bmod 2^{n/2} \text{ and}$$

$$f_2 :(e_{n/2+1}, \ldots, e_n) \mapsto t - \sum_{i=n/2+1}^{n} e_i a_i \bmod 2^{n/2}.$$

**Algorithm** Subset Sum with Low Memory

**Input:** $f_1, f_2$

1. **Repeat**
   1. Find a random collision $(x_1, x_2)$ in $f_1, f_2$
2. **Until** $\sum_{i=1}^{n/2} e_i a_i = t - \sum_{i=n/2+1}^{n} e_i a_i$

**Output: e**

# Analysis

**Run time:**

- We have $f_i : \{0,1\}^{\frac{n}{2}} \to \mathbb{Z}_{2^{\frac{n}{2}}}$ with search space size $|\mathcal{S}| = 2^{\frac{n}{2}}$.
- We expect that $f_1, f_2$ have $|S| = 2^{\frac{n}{2}}$ many collisions.
- Since we uniquely represent **e**, only a single collision is good.
- Need on expectation $\tilde{\mathcal{O}}(2^{\frac{n}{2}})$ iteration with cost $\tilde{\mathcal{O}}(2^{\frac{n}{4}})$ each.

**Exercise:** Generalize to solutions **e** with $\mathrm{wt}(\mathbf{e}) = \alpha$.

# 0.72 Algorithm (Becker, Coron, Joux 2011)

**Idea:**

- Represent $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ with $\mathbf{e}_1, \mathbf{e}_2 \in \{0,1\}^n$, $\mathrm{wt}(\mathbf{e}_i) = \frac{n}{4}$.
- Let $\mathcal{S} := \{\mathbf{e}' \in \{0,1\}^n \mid \mathrm{wt}(\mathbf{e}') = \frac{n}{4}\}$ with $|\mathcal{S}| = \binom{n}{n/4} \approx 2^{0.811n}$.
- Use collision finding in $f_1, f_2 : \mathcal{S} \to \mathbb{Z}_{|\mathcal{S}|}$ with

$$f_1 :(e_1, \ldots, e_n) \mapsto \sum_{i=1}^{n} e_i a_i \bmod 2^{0.811n} \text{ and}$$

$$f_2 :(e_1, \ldots, e_n) \mapsto t - \sum_{i=1}^{n} e_i a_i \bmod 2^{0.811n}.$$

**Algorithm** Subset Sum with Low Memory

**Input:** $f_1, f_2$

1. **Repeat**
   1. Find a random collision $(x_1, x_2)$ in $f_1, f_2$
2. **Until** $\sum_{i=1}^{n/2} e_i a_i = t - \sum_{i=n/2+1}^{n} e_i a_i \bmod 2^{0.811n}$

**Output:** $\mathbf{e}$

# Analysis

**Run Time:**

- There are $\binom{n/2}{n/4} = \tilde{\Theta}(2^{\frac{n}{2}})$ representations $\mathbf{e}$.
- Overall run time is

$$\tilde{\mathcal{O}}(|\mathcal{S}|) \cdot \frac{\binom{n/2}{n/4}}{|\mathcal{S}|} = \frac{|\mathcal{S}|^{\frac{3}{2}}}{\binom{n/2}{n/4}} = \tilde{\mathcal{O}}(2^{0.72n}).$$

**Remarks:**

- Hash function $h$ from DLP is now simply the ring homomorphism

$$\mathbb{Z}_{2^n} \to \mathbb{Z}_{2^{0.811n}},\ x \bmod 2^n \mapsto x \bmod 2^{0.811.n}.$$

- Hence subset sum allows more (subgroup) structure than DLP.
- Especially we can do a nested collision finding on the whole $\mathbb{Z}_{2^n}$.

**Theorem** Esser, May (2019)

Subset Sum can be solved in time $2^{0.65n}$ and space $\tilde{\mathcal{O}}(1)$.

# Howgrave-Graham Joux (2010)

**Idea:**

- Represent $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ with $\mathbf{e}_1, \mathbf{e}_2 \in \{0,1\}^n$, $\mathrm{wt}(\mathbf{e}_i) = \frac{n}{4}$.
- Let $\mathcal{S}_1 := \{\mathbf{e}' \in \{0,1\}^n \mid \mathrm{wt}(\mathbf{e}') = \frac{n}{4}\}$ with $|\mathcal{S}| = \binom{n}{n/4} \approx 2^{0.811n}$.
- We have $R_1 = \binom{n/2}{n/4}$ representations of $\mathbf{e}$. Then $\log R_1 \approx \frac{n}{2}$. Define

$$L_1 = \left\{ \sum_{i=1}^{n} e_i a_i \mid \mathbf{e} \in \mathcal{S}, \sum_{i=1}^{n} e_i a_i = 0 \bmod 2^{\frac{n}{2}} \right\},$$

$$L_2 = \left\{ t - \sum_{i=1}^{n} e_i a_i \mid \mathbf{e} \in \mathcal{S}, \sum_{i=1}^{n} t - e_i a_i = 0 \bmod 2^{\frac{n}{2}} \right\}.$$

- Then (on expectation) there exists a representation in $L_1 \times L_2$.
- We have $|L_1| = |L_2| = 2^{0.311n}$. Thus, we require at least time $2^{0.311n}$.
- **Observe:** Constructing $L_1, L_2$ is again a subset sum problem.

# Getting below $2^{\frac{n}{2}}$.

**Algorithm** Subset Sum 1

**Input:** $a_1, \dots, a_n, t$

1. Construct $L_1, L_2$ with Schroeppel-Shamir.
2. Compute $L = L_1 \bowtie_n L_2$.

**Output:** $L \cap \{0,1\}^n$

**Run Time:**

- Step 1 runs in time $2^{0.406n}$.
- We expect
$$|L| = \frac{|L_1| \cdot |L_2|}{2^{\frac{n}{2}}} = 2^{0.122n}.$$
- We can construct $L$ in time $\tilde{\mathcal{O}}(\max\{|L_1|, |L_2|, |L|\}) = 2^{0.311n}$.
- Therefore, we obtain total run time $2^{0.406n}$.

**Idea:** Construct $L_1, L_2$ recursively with algorithm Subset Sum 1.

# One more iteration

- We show how to construct $L_1$ ($L_2$ is analogous). Recall that $L_1 = \left\{ \sum_{i=1}^n e_i a_i \mid \mathbf{e} \in \{0,1\}^n, \mathrm{wt}(\mathbf{e}) = \frac{n}{4}, \sum_{i=1}^n e_i a_i = 0 \bmod 2^{\frac{n}{2}} \right\}$.
- Represent $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ with $\mathbf{e}_1, \mathbf{e}_2 \in \{0,1\}^n$, $\mathrm{wt}(\mathbf{e}_i) = \frac{n}{8}$.
- Let $\mathcal{S}_2 := \{\mathbf{e}' \in \{0,1\}^n \mid \mathrm{wt}(\mathbf{e}') = \frac{n}{8}\}$ with $|\mathcal{S}| = \binom{n}{n/8} \approx 2^{0.5435n}$.
- We have $R_2 = \binom{n/4}{n/8}$ representations of $\mathbf{e}$. Then $\log_2 R \approx \frac{n}{4}$. Define

$$L_1' = \left\{ \sum_{i=1}^n e_i a_i \mid \mathbf{e} \in \mathcal{S}_2, \sum_{i=1}^n e_i a_i = 0 \bmod 2^{\frac{n}{4}} \right\}.$$

- Then (on expectation) there exists a representation in $L_1' \times L_1'$.
- We expect that $|L_1'| = 2^{0.2935n}$.

# Getting to $2^{0.337n}$

**Algorithm** Howgrave-Graham Joux (2010)

**Input:** $a_1, \ldots, a_n, t$

1. Construct $L_1, L_2$ with Algorithm Subset Sum 1.
2. Construct
$L = (L_1 \times L_2) \cap \{\mathbf{e}' = \mathbf{e}_1 + \mathbf{e}_2 \mid \mathbf{e}_i \in \mathcal{S}, \sum_{i=1}^{n} e_i a_i = t \bmod 2^n\}$.

**Output:** $L \cap \{0, 1\}^n$

**Run Time:**

- Let $T$ be the size of $L_1, L_2$ before filtering out non-binary vectors.
- We expect $T = \frac{|L_1'| \cdot |L_1'|}{2^{\frac{n}{4}}} = 2^{2 \cdot 0.2935n - 0.25n} = 2^{0.337n}$.

**Theorem** Run Time of HGJ algorithm

HGJ solves subset sum instances in $2^{0.337n}$.

# The Becker-Coron-Joux algorithm (2011)

**Idea** of the BCJ algorithm:

- Represent $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ with $\mathbf{e}_i \in \{-1, 0, 1\}^n$.
- Advantage: Many more representations as in HGJ.
- Disadvantage: Also more sums that do not end up in $\{0, 1\}^n$.

**Theorem** Becker-Coron-Joux (2011)

BCJ solves subset sum in $2^{0.291n}$.

**Theorem** Esser, May (2019)

Subset Sum can be solved in $2^{0.255n}$.

- See https://arxiv.org/abs/1907.04295.
- Technique: Sampling instead of enumeration.

# Decoding of Linear Codes

**Definition** Linear code

A linear code $\mathcal{C}$ is a $k$-dimensional subspace of $\mathbb{F}_2^n$.

- We may define $\mathcal{C}$ via a generator matrix $G \in \mathbb{F}_2^{k \times n}$:

$$\mathcal{C} = \{\mathbf{c} = \mathbf{m}G \in \mathbb{F}_2^n \mid \mathbf{m} \in \mathbb{F}_2^k\}$$

- Let $d = \min_{\mathbf{c},\mathbf{c}'}\{\mathrm{wt}(\mathbf{c} + \mathbf{c}')\}$ be the distance.
- Let $\mathbf{x} = \mathbf{c} + \mathbf{e} \in \mathbb{F}_2^n$ with $\mathbf{c} \in \mathcal{C}$, $\mathrm{wt}(\mathbf{e}) \leq \frac{d-1}{2}$.
- Then $\mathbf{x}$ can uniquely be decoded to $\mathbf{c}$.

**Definition** Decoding Problem

**Given:** $G$, $\mathbf{x} = \mathbf{c} + \mathbf{e} \in \mathbb{F}_2^n$ with $\mathbf{c} \in \mathcal{C}$, $\mathrm{wt}(\mathbf{e}) \leq \frac{d-1}{2}$
**Find:** $\mathbf{c}$ (or equivalently $\mathbf{e}$)

# Parity Check Matrix

- Alternatively, we may define $\mathcal{C}$ via some parity check $P \in \mathbb{F}_2^{(n-k)\times n}$:

$$\mathcal{C} = \{\mathbf{c} \in \mathbb{F}_2^n \mid P\mathbf{c} = \mathbf{0}\}.$$

- By linearity we have $P\mathbf{x} = P(\mathbf{c} + \mathbf{e}) = P\mathbf{c} + P\mathbf{e} = P\mathbf{e}$.
- Let us call $\mathbf{s} = P\mathbf{x}$ the syndrome of $\mathbf{x}$.
- Then we have to find a minimal weight $\mathbf{e}$ satisfying $P\mathbf{e} = \mathbf{s}$.

**Syndrome decoding**

**Given:** $P$, $\mathbf{x} = \mathbf{c} + \mathbf{e} \in \mathbb{F}_2^n$ with $\mathbf{c} \in \mathcal{C}$, $\mathrm{wt}(\mathbf{e}) \leq \frac{d-1}{2}$

**Find:** $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) \leq \frac{d-1}{2}$ satisfying $P\mathbf{e} = P\mathbf{x}$.

- $P\mathbf{e} = \mathbf{s}$ is a subset sum problem in $(\mathbb{F}_2^n, +)$ (instead of $(\mathbb{Z}_{2^n}, +)$).

# Relation to McEliece

- Typical McEliece parameters: $k = 0.8n, \mathrm{wt}(\mathbf{e}) = \omega = 0.02n$.
- **Question:** For which $n$ do we get 80-bit security?

**Brute-Force Decoding**

**Input:** $P, \mathbf{s}$

1. **For** all $\mathbf{e} \in \mathbb{F}_2^n$ with $\omega = 0.02n$
    1. **If** $(P\mathbf{e} = \mathbf{s})$ output $\mathbf{e}$.

**Output: e**

**Run Time:**

- Search space size $\binom{n}{0.02n} = 2^{H(0.02)n} = 2^{0.14n}$.
- We have $H(0.02)n \geq 80$ for $n \geq 566$.

# Meet-in-the-Middle Syndrome Decoding

**Meet-in-the-Middle:**

- Split $\mathbf{e} = \mathbf{e}_1 || \mathbf{e}_2$ with $\mathbf{e}_i \in \mathbb{F}_2^{\frac{n}{2}}$, $\mathrm{wt}(\mathbf{e}_i) = 0.01n$.
- Split $P = (P_1 | P_2)$ with $P_i \in \mathbb{F}_2^{0.2n \times n/2}$.
- MitM equation is $P_1 \mathbf{e}_1 = \mathbf{s} + P_2 \mathbf{e}_2$.
- Search space is $\binom{n/2}{0.02 \cdot n/2} = 2^{H(0.02)\frac{n}{2}} = 2^{0.07n}$.
- We have $H(0.02)\frac{n}{2} \geq 80$ for $n \geq 1132$.

# Adaption of Howgrave-Graham Joux

**Idea:**

- Represent $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ with $\mathbf{e}_i \in \mathbb{F}_2^n$, $\mathrm{wt}(\mathbf{e}_i) = 0.01$.
- Let $\mathcal{S} := \{\mathbf{e}' \in \{0,1\}^n \mid \mathrm{wt}(\mathbf{e}') = 0.01\}$ with $|\mathcal{S}| = \binom{n}{0.01n} \approx 2^{H(0.01)n}$.
- We have $R = \binom{0.02n}{0.01n}$ representations of $\mathbf{e}$. Then $\log R_1 \approx 0.02n$. Define

$$L_1 = \left\{ P_1\mathbf{e}_1 \in \{0\}^{0.02} \times \mathbb{F}_2^{0.18n} \mid \mathbf{e}_1 \in \mathcal{S} \right\},$$
$$L_2 = \left\{ \mathbf{s} + P_2\mathbf{e}_2 \in \{0\}^{0.02} \times \mathbb{F}_2^{0.18n} \mid \mathbf{e}_2 \in \mathcal{S} \right\}.$$

- Then (on expectation) there exists a representation in $L_1 \times L_2$.
- We have $|L_1| = |L_2| = 2^{(H(0.01)-0.02)n} = 2^{0.061n}$.

**Algorithm** (Not yet) May, Meurer, Thomae

**Input:** $P, \mathbf{s}$

1. Construct $L_1, L_2$ with Meet-in-the-Middle.
2. Construct
   $L = L_1 \bowtie_{0.2n} L_2 = \{\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 \mid \mathrm{wt}(\mathbf{e}) = 0.02n, P\mathbf{e} = s \in \mathbb{F}_2^{0.2n}\}$.

**Output:** $L \cap \{0,1\}^n$

**Run Time:**

- Step 1: Meet-in-the-Middle has input list sizes $2^{\frac{H(0.01)}{2}n} \approx 2^{0.04n}$.
- Output list sizes are $|L_1| = |L_2| = 2^{(H(0.01)-0.02)n} = 2^{0.061n}$.
- Thus, step 1 runs in time $2^{0.061n}$.
- We expect that $|L| = 1$, since decoding has a unique solution.
- Therefore, we obtain total run time $2^{0.061n}$.
- We have $(H(0.01) - 0.02)n \geq 80$ for $n \geq 1316$.

**Exercise:** Do *Information Set Decoding.*